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Remarks on the Periodic Boundary Value Problems for First-Order Differential Equations

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Abstract—This paper is concerned with the existence of solutions and the monotone method of first-order periodic boundary value problems when the lower and upper solutions α and β violate the boundary conditions $\alpha(0) \leq \alpha(2\pi)$ and $\beta(0) \geq \beta(2\pi)$. Using the topological degree theory, two existence theorems are established under weaker conditions than the one-side Lipschitz conditions. An example is given, which illustrates that PBVP may not have solutions between α and β without further restrictions to $f(t,u)$. The monotone method is also discussed with some new results. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

We are concerned with the existence of solutions and the monotone method for periodic boundary value problems (PBVP) of first-order differential equations of the form

$$u' = f(t,u), \qquad u(0) = u(2\pi), \tag{1.1}$$

where $f \in C[I \times R, R]$, $I = [0, 2\pi]$. This problem has been studied by several authors [1–4]. When the lower and upper solutions α and β for PBVP (1.1) satisfy

$$\alpha(0) \leq \alpha(2\pi), \qquad \beta(0) \geq \beta(2\pi), \tag{1.2}$$

the existence of solutions was proved by using an abstract existence theorem [1, 5]. Furthermore, the extreme solutions were obtained as limits of monotone iterates if adding one-side Lipschitz condition to $f(t,u)$. For the case that α and β violate the relations of (1.2), only the monotone iterative method for extreme solutions was discussed and the existence of solutions between the lower and upper solutions was left as an open problem [2, 3]. In this paper, by using the

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topological degree theory, we will show two existence theorems under weaker conditions than the one-side Lipschitz condition. We will also give an example to illustrate that (1.1) may not have solutions between the lower and upper solutions without further restrictions to $f(t,u)$. In the last section, we will discuss the monotone iteration for (1.1). Our results and the results in [2,3] do not include each other.

The following two comparison results are useful in the sequel.

LEMMA 1.1. *Let $m \in C^1[I,R]$ be such that*

$$m' \le -Mm - \gamma_m, \qquad t \in I,$$

where $M > 0$ is a constant and

$$\gamma_m = \begin{cases} 0, & \text{if } m(0) \le m(2\pi), \\ \dfrac{Mt+1}{2\pi}[m(0) - m(2\pi)], & \text{if } m(0) > m(2\pi). \end{cases}$$

Then $m(t) \le 0$ on I.

PROOF. The case for $m(0) \le m(2\pi)$ was given in [1]. To prove the case for $m(0) > m(2\pi)$, let $m_1(t) = m(t) + (t/2\pi)[m(0) - m(2\pi)]$. Then $m_1(0) = m_1(2\pi)$ and $m_1'(t) \le -Mm_1(t)$. Hence, $m_1(t) \le 0$ on I, and therefore $m(t) \le 0$ on I.

LEMMA 1.2. *Let $m \in C^1[I,R]$ be such that*

$$m' \ge Mm + \gamma_m, \qquad t \in I,$$

where $M > 0$ is a constant and

$$\gamma_m = \begin{cases} 0, & \text{if } m(0) \ge m(2\pi), \\ \dfrac{M(2\pi - t)+1}{2\pi}[m(2\pi) - m(0)], & \text{if } m(0) < m(2\pi). \end{cases}$$

Then $m(t) \le 0$ on I.

PROOF. Let $m_2(t) = m(2\pi - t)$, then $m_2(t)$ satisfies the conditions of Lemma 1.1. Hence, $m_2(t) \le 0$ on I, and therefore $m(t) \le 0$ on I.

2. EXISTENCE THEOREMS

Employing the topological degree theory to the modified problems, we can prove the following existence theorems of solutions between the lower and upper solutions. With respect to the lower and upper solutions α and β, as well as the function $f(t,u)$ in the right-hand side of (1.1), we list the following assumptions.

(A_0)(i) $\alpha, \beta \in C^1[I,R]$, $\alpha(t) \le \beta(t)$, $t \in I$.

(A_0)(ii) $\alpha, \beta \in C^1[I,R]$, $\beta(t) \le \alpha(t)$, $t \in I$.

(A_1)(i) $\alpha' \le f(t,\alpha) - \gamma_\alpha$, $\beta' \ge f(t,\beta) + \gamma_\beta$, where

$$\gamma_\alpha = \begin{cases} 0, & \text{if } \alpha(0) \le \alpha(2\pi), \\ \dfrac{Mt+1}{2\pi}[\alpha(0) - \alpha(2\pi)], & \text{if } \alpha(0) > \alpha(2\pi), \end{cases}$$

$$\gamma_\beta = \begin{cases} 0, & \text{if } \beta(0) \ge \beta(2\pi), \\ \dfrac{mt+1}{2\pi}[\beta(2\pi) - \beta(0)], & \text{if } \beta(0) < \beta(2\pi). \end{cases}$$

(A_1)(ii) $\alpha' \le f(t,\alpha) - \gamma_\alpha$, $\beta' \ge f(t,\beta) + \gamma_\beta$, where

$$\gamma_\alpha = \begin{cases} 0, & \text{if } \alpha(0) \le \alpha(2\pi), \\ \dfrac{M(2\pi - t)+1}{2\pi}[\alpha(0) - \alpha(2\pi)], & \text{if } \alpha(0) > \alpha(2\pi), \end{cases}$$

$$\gamma_\beta = \begin{cases} 0, & \text{if } \beta(0) \ge \beta(2\pi), \\ \dfrac{M(2\pi - t)+1}{2\pi}[\beta(2\pi) - \beta(0)], & \text{if } \beta(0) < \beta(2\pi). \end{cases}$$

(A_2)(i) f satisfies the following conditions relative to α, β: when $\alpha(0) > \alpha(2\pi)$,

$$f(t,u) - f(t,\alpha(t)) \geq -M(u - \alpha(t)),$$

whenever $t \in I$, $\alpha(t) \leq u \leq \beta(t)$; when $\beta(0) < \beta(2\pi)$,

$$f(t,\beta(t)) - f(t,u) \geq -M(\beta(t) - u),$$

whenever $t \in I$, $\alpha(t) \leq u \leq \beta(t)$.

(A_2)(ii) f satisfies the following conditions relative to α, β: when $\alpha(0) > \alpha(2\pi)$,

$$f(t,\alpha(t)) - f(t,u) \leq M(\alpha(t) - u),$$

whenever $t \in I$, $\beta(t) \leq u \leq \alpha(t)$; when $\beta(0) < \beta(2\pi)$,

$$f(t,u) - f(t,\beta(t)) \leq M(u - \beta(t)),$$

whenever $t \in I$, $\beta(t) \leq u \leq \alpha(t)$.

THEOREM 2.1. *Let (A_0)(i), (A_1)(i), and (A_2)(i) hold. Then there exists a solution u of PBVP (1.1) such that $\alpha(t) \leq u(t) \leq \beta(t)$ on I.*

We leave out the proof of this theorem because it can be completed in the same way as the proof of the following theorem.

THEOREM 2.2. *Let (A_0)(ii), (A_1)(ii), and (A_2)(ii) hold. Then there exists a solution u of PBVP (1.1) such that $\beta(t) \leq u(t) \leq \alpha(t)$ on I.*

PROOF. Let us consider the following modified problem relative to PBVP (1.1):

$$\begin{aligned} u' &= f(t,p(t,u(t))) + M(u(t) - p(t,u(t))), \\ u(0) &= u(2\pi), \end{aligned} \tag{2.1}$$

where $p : I \times R \to R$ is defined by

$$p(t,u) = \begin{cases} \alpha(t), & \text{for } u > \alpha(t), \\ u, & \text{for } \beta(t) \leq u \leq \alpha(t), \\ \beta(t), & \text{for } u < \beta(t). \end{cases}$$

Then, the proof is completed by the following three lemmas.

LEMMA 2.1. *If $u \in C^1[I,R]$ be such that $\beta(t) \leq u(t) \leq \alpha(t)$ on I, then u is a solution of (1.1) if and only if it is a solution of (2.1).*

LEMMA 2.2. *Every solution $u(t)$ of (2.1) satisfies $\beta(t) \leq u(t) \leq \alpha(t)$ on I.*

LEMMA 2.3. *The modified problem (2.1) possesses at least one solution $u(t)$.*

Lemma 2.1 is obvious.

The proof of Lemma 2.2 can be completed as follows. Let $u \in C^1[I,R]$ be a solution of (2.1). We shall only prove $u \leq \alpha$ since proving $\beta \leq u$ will be similar. We need to discuss the following two cases:

(1) $\alpha(0) \leq \alpha(2\pi)$, and
(2) $\alpha(0) > \alpha(2\pi)$.

(1) For $\alpha(0) \le \alpha(2\pi)$, let $m(t) = u(t) - \alpha(t)$, then $m(0) \ge m(2\pi)$. Define $D = \{t \in I : m(t) > 0\}$, then

$$\begin{aligned} m'(t) &= u'(t) - \alpha'(t) \\ &\ge f(t, p(t, u(t))) + M(u(t) - p(t, u(t))) - f(t, \alpha(t)) \\ &> 0, \qquad \text{for } t \in D, \end{aligned} \tag{2.2}$$

because of condition (A_1)(ii). Hence, there exists $t_0 \in I$ such that $m(t_0) \le 0$. In fact, if this were not sure, then $m(t) > 0$ on I. Then $D = I$, we have $m'(t) > 0$ on I by (2.2). This implies $m(0) < m(2\pi)$, we get a contradiction. Furthermore, we can prove $m(2\pi) \le m(0) \le 0$. Indeed, if $m(0) > 0$, then there exists $t_1 \in (0, 2\pi]$ such that $m(t_1) = 0$ and $m(t) > 0$ on $[0, t_1)$. Then $m(0) < m(t_1) = 0$ by (2.2), this is a contradiction again. Finally, we show that $m(t) \le 0$, i.e., $u(t) \le \alpha(t)$ on I. If it were false, then there exists $t^* \in (0, 2\pi)$ such that $m(t^*) = \max_I m(t) > 0$. This implies that there exists $t_2 \in (t^*, 2\pi]$ such that $m(t_2) = 0$ and $m(t) > 0$ for $t \in [t^*, t_2)$. It yields that $m'(t) > 0$ on $[t^*, t_2)$ by (2.2). Therefore $m(t^*) < m(t_2) = 0$ and that leads to a contradiction.

(2) For $\alpha(0) > \alpha(2\pi)$, let $m(t) = u(t) - \alpha(t)$, then we have $m(0) < m(2\pi)$ and

$$\begin{aligned} m'(t) &= u'(t) - \alpha'(t) \\ &\ge f(t, p(t, u(t))) + M(u(t) - p(t, u(t))) - f(t, \alpha(t)) \\ &\quad + \frac{M(2\pi - t) + 1}{2\pi}[m(2\pi) - m(0)] \\ &\ge Mm(t) + \frac{M(2\pi - t) + 1}{2\pi}[m(2\pi) - m(0)] \end{aligned}$$

by (A_1)(ii) and (A_2)(ii). Hence, $m(t) \le 0$ on I by Lemma 1.2, that is, $u(t) \le \alpha(t)$ on I.

We now show the proof of Lemma 2.3. Consider the following family of periodic boundary value problems:

$$\begin{aligned} u'(t) - Mu(t) &= \lambda(f(t, p(t, u(t))) - Mp(t, u(t)))x, \\ u(0) &= u(2\pi), \end{aligned} \tag{2.3}$$

where $\lambda \in [0, 1]$. Set

$$X = \{u \in C^1[I, R] : u(0) = u(2\pi)\},$$

then X with the norm $\|.\|_{C^1}$ is a Banach space. Define the operator $L : X \to C[I, R]$ by $Lu = u' - Mu$ and the operator $N : C[I, R] \to C[I, R]$ by

$$Nu = f(t, p(t, u(t))) - Mp(t, u(t)).$$

It is easy to see that N and L are continuous. The linear operator L is one-to-one and onto since $u(0) = u(2\pi)$. It follows, by the open mapping theorem, that $J = L^{-1}$ is also continuous. So (2.3) is transformed into the abstract equation

$$Lu = \lambda Nu, \qquad \lambda \in [0, 1], \quad u \in X, \tag{2.4}$$

and the equation (2.4) is equivalent to

$$u = \lambda JNu, \qquad \lambda \in [0, 1], \quad u \in X, \tag{2.5}$$

where $JN : X \to X$ is continuous compact operator since if $B \subset X$ is bounded, then B is relatively compact in $C[I, R]$.

Let $\mu = \max(\max_I |\alpha(t)|, \max_I |\beta(t)|)$ and suppose that $|f(t, u)| \le P$ for $t \in I$, $-\mu \le u \le \mu$. Then, we have

$$\|\lambda Nu\|_C \le P + M\mu, \qquad \text{for } u \in C[I, R], \quad \lambda \in [0, 1].$$

Therefore, for any solution $u \in X$ of (2.5), we have

$$\|u\|_X \le \|J\| \|\lambda N u\|_C \le \|J\|(P + M\mu).$$

Hence, $(id - \lambda JN)u \neq 0$ for $(\lambda, u) \in [0,1] \times \partial B_r(0)$, where the radius r of the ball $B_r(0)$ is such that $r > (P + M\mu)\|J\|$. Now, the homotopy invariance of Leray-Schauder degree [6] yields

$$D(id - JN, B_r(0), 0) = D(id, B_r(0), 0) = 1.$$

Hence, (2.5) with $\lambda = 1$, or equivalently (2.1), possesses at least one solution $u(t)$ in X. The proof of the theorem is now completed.

The following example illustrates that (1.1) may not have solutions between the lower and upper solutions without further restrictions to $f(t,u)$.

EXAMPLE 2.1. Consider the following PBVP:

$$u' = f(t,u) = u, \qquad u(0) = u(2\pi). \tag{2.6}$$

It is easy to check that $\alpha(t) \equiv 1$ is a lower solution of (2.6), and $\beta(t) = e^t - (e^{2\pi} - 1/2)$ a upper solution with $M = 1 - 1/2\pi$. In fact, $\beta(0) = 3/2 - e^{2\pi} < 0 < 1/2 = \beta(2\pi)$, and for $t \in [0, 2\pi]$,

$$f(t,\beta(t)) + \frac{M(2\pi - t) + 1}{2\pi}[\beta(2\pi) - \beta(0)] \le e^t - \left(e^{2\pi} - \frac{1}{2}\right) + \left(e^{2\pi} - 1\right) < e^t = \beta'(t).$$

Furthermore, we have $\beta(t) \le \alpha(t)$ on $[0, 2\pi]$. Hence, the conditions (A_0)(ii) and (A_1)(ii) of Theorem 2.2 hold, but (2.6) has no solution between this pair of lower and upper solutions since (2.6) only has one solution $u \equiv 0$ which does not satisfy $\beta \le u \le \alpha$.

It is easy to see that $\alpha(t) \equiv 1$ and $\beta(t) = e^t - (e^{2\pi} + e^{2\pi}/2\pi)$ are also lower and upper solutions with $M = 1$ of (2.6) and that $f(t,u) = u$ satisfies (A_2)(ii) with $M = 1$. Hence, by Theorem 2.2, (2.6) has a solution u between this pair of lower and upper solutions, that is, $u \equiv 0$.

3. MONOTONE ITERATION

If the condition (A_2) in Section 2 is strengthened into the following one-side Lipschitz conditions, then the extremal solutions can be obtained as the limits of monotone iterative sequences.

(A_2')(i) $f(t,u_1) - f(t,u_2) \ge -M(u_1 - u_2)$ whenever $t \in I$, $\alpha(t) \le u_2 \le u_1 \le \beta(t)$.
(A_2')(ii) $f(t,u_1) - f(t,u_2) \le M(u_1 - u_2)$ whenever $t \in I$, $\beta(t) \le u_2 \le u_1 \le \alpha(t)$.

THEOREM 3.1. *Let (A_0)(i), (A_1)(i), and (A_2')(i) hold. Then there exist monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_1 = \alpha$ and $\beta_1 = \beta$, such that we have $\lim_{n\to\infty} \alpha_n(t) = \rho(t)$ and $\lim_{n\to\infty} \beta_n(t) = \gamma(t)$ uniformly on I, and that ρ and γ are minimal and maximal solutions of (1.1) in $[\alpha, \beta] = \{u \in C[I,R] : \alpha \le u \le \beta\}$, respectively.*

We leave out the proof of this theorem because it can be completed in the same way as the proof of following theorem.

THEOREM 3.2. *Let (A_0)(ii), (A_1)(ii), and (A_2')(ii) hold. Then there exist monotone sequences $\{\beta_n\}$ and $\{\alpha_n\}$ with $\beta_1 = \beta$ and $\alpha_1 = \alpha$, such that we have $\lim_{n\to\infty} \beta_n(t) = \rho(t)$ and $\lim_{n\to\infty} \alpha_n(t) = \gamma(t)$ uniformly on I, and that ρ and γ are minimal and maximal solutions of (1.1) in $[\beta, \alpha] = \{u \in C[I,R] : \beta \le u \le \alpha\}$, respectively.*

PROOF. For any $\eta \in [\beta, \alpha]$, we consider the following linear PBVP:

$$\begin{aligned} u' &= f(t,\eta(t)) + M(u - \eta(t)), \\ u(0) &= u(2\pi). \end{aligned} \tag{3.1}$$

Setting $\sigma = f(t, \eta) - M\eta$, we see that

$$u(t) = u(0)e^{Mt} + \int_0^t \sigma(s)e^{M(t-s)}\,ds,$$

where

$$u(0) = \frac{1}{e^{-2M\pi} - 1}\int_0^{2\pi} \sigma(s)e^{-Ms}\,ds$$

is the unique solution of (3.1). We define a mapping A by $A\eta = u$ for $\eta \in [\beta, \alpha]$, where u is the unique solution of (3.1). Then it can be shown by a repeated application of Lemma 1.2 that $\beta \leq A\beta$, $A\alpha \leq \alpha$, and A is monotone nondecreasing on $[\beta, \alpha]$. Define $\{\beta_n\}$, $\{\alpha_n\}$ with $\beta_1 = \beta$, $\alpha_1 = \alpha$ by

$$\beta_n = A\beta_{n-1}, \qquad \alpha_n = A\alpha_{n-1}.$$

Then we have

$$\beta = \beta_1 \leq \cdots \leq \beta_n \leq \cdots \leq \alpha_n \leq \cdots \leq \alpha_1 = \alpha, \qquad t \in I.$$

It then follows, by using standard arguments [2], that $\lim_{n\to\infty} \beta_n(t) = \rho(t)$ and $\lim_{n\to\infty} \alpha_n(t) = \gamma(t)$ uniformly and monotonically on I, and that ρ and γ are minimal and maximal solutions of (1.1), respectively. The proof is complete.

In [2,3], it was proved that Theorems 3.1 and 3.2 still hold if (A_1)(i) and (A_1)(ii) are replaced by

(A_1') $\alpha' \leq f(t, \alpha) - \gamma_\alpha$, $\beta' \geq f(t, \beta) + \gamma_\beta$, where

$$\gamma_\alpha = \begin{cases} 0, & \text{if } \alpha(0) \leq \alpha(2\pi), \\ \dfrac{Me^{2M\pi}}{e^{2M\pi} - 1}[\alpha(0) - \alpha(2\pi)], & \text{if } \alpha(0) > \alpha(2\pi), \end{cases}$$

$$\gamma_\beta = \begin{cases} 0, & \text{if } \beta(0) \geq \beta(2\pi), \\ \dfrac{Me^{2M\pi}}{e^{2M\pi} - 1}[\beta(2\pi) - \beta(0)], & \text{if } \beta(0) < \beta(2\pi), \end{cases}$$

respectively.

The following examples illustrate that these results and Theorems 3.1 and 3.2 do not include each other.

EXAMPLE 3.1. Consider the following PBVP:

$$u' = f(t, u) = -u, \qquad u(0) = u(2\pi). \tag{3.2}$$

Now, $\alpha(t) = -t$ and $\beta(t) = t$ satisfy (A_0)(i) and (A_1)(i) with $M = 1$. In fact, $\alpha(0) = 0 > -2\pi = \alpha(2\pi)$ and

$$f(t, \alpha(t)) - \frac{Mt+1}{2\pi}[\alpha(0) - \alpha(2\pi)] = t - \frac{t+1}{2\pi}2\pi = -1 = \alpha'(t), \qquad \text{for } t \in I.$$

Also, $\beta(0) = 0 < 2\pi = \beta(2\pi)$ and

$$f(t, \beta(t)) + \frac{Mt+1}{2\pi}[\beta(2\pi) - \beta(0)] = -t + \frac{t+1}{2\pi}2\pi = 1 = \beta'(t), \qquad \text{for } t \in I.$$

Clearly, (A_2)(i) with $M = 1$ holds. Hence, starting from α and β, we can use Theorem 3.1 to obtain the approximated sequences of the solution of (3.2). But this result cannot be obtained by using the corresponding theorems in [2,3] since these α and β do not satisfy (A_1') with $M = 1$.

EXAMPLE 3.2. Consider (2.6) once more. $\alpha(t) \equiv 1$ and $\beta(t) = e^t - e^{2\pi}$ satisfy (A_1') with $M = 1$. Indeed $\beta(0) = 1 - e^{2\pi} < 0 = \beta(2\pi)$ and

$$f(t, \beta(t)) + \frac{Me^{2M\pi}}{e^{2M\pi} - 1}[\beta(2\pi) - \beta(0)] = e^t - e^{2\pi} + \frac{e^{2\pi}}{e^{2\pi} - 1}\left(e^{2\pi} - 1\right) = e^t = \beta'(t),$$

for $t \in I$. But it is easy to check that α and β do not satisfy (A_1)(ii) with $M = 1$. Hence, starting from these α and β, we can use corresponding theorems in [2,3] to obtain the approximated sequences of solution of (2.6), but this result cannot be obtained by Theorem 3.2.

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